Probing the tails of the ground-state energy distribution for the directed polymer in a random medium of dimension $d=1,2,3$ via a Monte Carlo procedure in the disorder

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(Received 18 July 2006; published 13 November 2006)

In order to probe with high precision the tails of the ground-state energy distribution of disordered spin systems, Körner, Katzgraber, and Hartmann have recently proposed an importance-sampling Monte Carlo Markov chain in the disorder. In this paper, we combine their Monte Carlo procedure in the disorder with exact transfer matrix calculations in each sample to measure the negative tail of ground-state energy distribution $P_d(E_0)$ for the directed polymer in a random medium of dimension $d=1,2,3$. In $d=1$, we check the validity of the algorithm by a direct comparison with the exact result, namely, the Tracy-Widom distribution. In dimensions $d=2$ and $d=3$, we measure the negative tail up to ten standard deviations, which correspond to probabilities of order $P_d(E_0) \sim 10^{-22}$. Our results are in agreement with Zhang's argument, stating that the negative tail exponent $\eta(d)$ of the asymptotic behavior $\ln P_d(E_0) \sim -|E_0|^{\eta(d)}$ as $E_0 \to -\infty$ is directly related to the fluctuation exponent $\theta(d)$ [which governs the fluctuations $\Delta E_0(L) \sim L^{\theta(d)}$ of the ground-state energy E_0 for polymers of length L] via the simple formula $\eta(d)=1/[1-\theta(d)]$. Throughout the paper, we comment on the similarities and differences with spin glasses.

DOI: 10.1103/PhysRevE.74.051109 PACS number(s): 02.50.Ng, 05.40.−a

I. INTRODUCTION

Since the ground-state energy E_0 of a disordered sample is the minimal energy among the energies of all possible configurations, the study of its distribution belongs to the field of extreme value statistics. Whereas the case of independent random variables is well classified in three universality classes [1], the problem for the correlated energies within a disordered sample remains open and has been the subject of many recent studies. The interest lies in both (i) the scaling behavior of the average $E_0^{\rm av}(L)$ and the standard deviation $\Delta E_0(L)$ with the size L and (ii) the asymptotic distribution $P(x)$ of the rescaled variable $x=[E_0-E_0^{\rm av}(L)]/\Delta E_0(L)$ in the limit $L \to \infty$

$$\mathcal{P}_L(E_0) \underset{L\to\infty}{\simeq} \frac{1}{\Delta E_0(L)} P\left(x = \frac{E_0 - E_0^{\rm av}(L)}{\Delta E_0(L)} \right). \tag{1}$$

We first recall what is known in the field of spin glasses, before focusing on the directed polymer model.

A. Ground-state energy distribution in spin glasses

For spin glasses in dimension d, let us consider samples containing $N=L^d$, where L denotes the linear size, and follow the notations of Ref. [2]. The "shift exponent" θ_s governs the correction to extensivity of the averaged value

$$E_0^{\rm av}(L) \sim L^d e_0 + L^{\theta_s} e_1 + \cdots = N e_0 + N^{\theta_s/d} e_1 + \cdots . \tag{2}$$

Within the droplet theory [3,4], this shift exponent θ_s coincides with the domain wall exponent $\theta_{\rm DW}$ and with the droplet exponent θ of low energy excitations. The "fluctuation exponent" θ_f governs the growth of the standard deviation

$$\Delta E_0(L) \sim L^{\theta_f} e_2 = N^{\theta_f/d} e_2 . \tag{3}$$

In any finite dimension d, it has been proven that the fluctuation exponent is $\theta_f=d/2$ [5]. Accordingly, the rescaled distribution $P(x)$ of Eq. (1) was numerically found to be Gaussian in $d=2$ and $d=3$ [2], suggesting some central limit theorem. On the contrary, in mean-field spin glasses, the width does not grow as $N^{1/2}$ and the distribution is not Gaussian. In the random energy model [6], the width remains finite $\Delta E_0(N) \sim O(1)$ and the distribution is the Gumbel distribution [7]. In the Sherrington-Kirpatrick model, the width grows as $\Delta E_0(N) \sim N^{1/4}$ according to some theoretical arguments [2,8] and numerics [2,9], and the distribution is clearly asymmetric [9,10]. Finally for the one-dimensional disordered spin chain with power-law interactions that allows one to interpolate between effectively finite-dimensional and mean-field models, the transition between short-range and infinite-range behaviors corresponds to the Gaussian–non-Gaussian transition for the ground-state energy [11].

B. Ground-state energy for the directed polymer

The directed polymer model in $1+d$ dimensions is defined by the following recursion for the partition function:

$$Z_{L+1,T}(\vec{r}) = \sum_{j=1}^{2d} e^{-\beta \epsilon_L(\vec{r}+\vec{e}_j,\vec{r})} Z_{L,T}(\vec{r}+\vec{e}_j), \tag{4}$$

where $\vec{e}_j$, $1 \leqslant j \leqslant 2d$ are the unit vectors linking a vertex and its $2d$ neighbors on the d-dimensional hypercubic lattice. The bond energies $\epsilon_L(\vec{r}+\vec{e}_j,\vec{r})$ are random independent variables, drawn with the Gaussian distribution

$$\rho(\epsilon) = \frac{1}{\sqrt{2\pi}} e^{-\epsilon^2/2}. \tag{5}$$

The full partition function $Z_L(T)$ at temperature $T=1/\beta$ for a polymer of length L

$$Z_{L,T} = \sum_{\vec{r}} Z_{L,T}(\vec{r}) \tag{6}$$

is simply the sum over all possible end points $\vec{r}$ that can be reached in "time" L by a random walk on the hypercubic lattice starting at the origin at time $L=0$, i.e., the transfer matrix [Eq. (4)] is iterated from the initial condition $Z_{L=0,T}(\vec{r}) = \delta_{\vec{r},\vec{0}}$. The free-energy $F(L,T)$ and the energy $E(L,T)$ are then defined by the usual definitions $F(L,T) = -T \ln Z_{L,T}$ and $E(L,T) = -\partial_\beta \ln Z_{L,T}$.

This model has attracted a lot of attention for two main reasons: (i) It is directly related to nonequilibrium properties of growth models [12] and (ii) as a disordered system, it presents some similarities with the spin-glass physics [3,12–15]. At low temperature, there exists a disorder dominated phase, where the order parameter is an "overlap" [13,15].

The probability distribution of the ground-state energy $E_0(L)$ is expected to follow the scaling form of Eq. (1). In contrast with spin glasses where the shift exponent θ_s [Eq. (2)] and the fluctuation exponent θ_f [Eq. (3)] are different, there is a single exponent $\theta(d)$ that governs both the correction to extensivity of the average $E_0^{\rm av}(L)$ and the width $\Delta E_0(L)$

$$E_0^{\rm av}(L) \sim L e_0 + L^{\theta(d)} e_1 + \cdots, \tag{7}$$

$$\Delta E_0(L) \sim L^{\theta(d)} e_2 + \cdots. \tag{8}$$

This exponent also governs the statistics of low excitations within the droplet theory [3], as confirmed numerically [16]. This exponent is exactly known in one dimension [17–20]

$$\theta(d=1) = 1/3 \tag{9}$$

and has been numerically measured in dimensions $d=2,3,4,5$ [16,22–25]

$$\theta(d=2) \sim 0.244, \tag{10}$$

$$\theta(d=3) \sim 0.186. \tag{11}$$

For the mean-field version on the Cayley tree, the exponent vanishes $\theta(d=\infty)=0$ [13,26], with a width of order $O(1)$ for the probability distribution, but with a nonrandom $O(\ln L)$ correction to the extensive term $e_0 L$ in the averaged value [13].

The rescaled distribution P_d is exactly known in $d=1$ and is related to Tracy-Widom distributions of the largest eigenvalue of random matrices ensembles [19–21]. On the Cayley tree, the rescaled distribution was found to be nonuniversal and to depend on the disorder distribution [26].

C. Numerical measure of the ground-state energy distribution

The numerical measure of the ground-state energy distribution is usually done by a simple sampling procedure, where the histogram of the energies of independent samples are collected. However, recently, Körner, Katzgraber, and Hartmann [10] have proposed an importance-sampling Monte Carlo algorithm in the disorder, which allows one to measure much more precisely the tails of the distribution. In the case of the Sherrington-Kirkpatrick model of spin glasses, this procedure was used to measure the negative tail on systems of size $N \leqslant 128$ [10] up to $x \geqslant -15$ corresponding to probabilities $P(x) \geqslant 10^{-18}$ [see Eq. (1)], whereas the simple sampling procedure cannot go beyond $x \geqslant -5$ corresponding to $P(x) \geqslant 10^{-4}$ [9].

For the directed polymer in dimensions $d=2$ and $d=3$, the rescaled distribution P_d has been numerically measured via simple sampling in Ref. [27] with results in the region $x \geqslant -5$. In this paper, we use the importance-sampling algorithm recently proposed in Ref. [10] to measure precisely the negative tail of the probability distribution up to $x \geqslant -10$.

The paper is organized as follows. In Sec. II, we recall Zhang's argument [12] that relates the decay of the rescaled distribution P_d to the fluctuation exponent θ_f. In Sec. III, we describe the Monte Carlo procedure in the disorder proposed in Ref. [10] and mention the specific choices for the application to the directed polymer model. In Sec. IV, we show the validity of the procedure in $d=1$ via the direct comparison with the exactly known distribution (Tracy-Widom). Finally in Secs. V and VI, we present our results for $d=2$ and $d=3$, respectively. We present our conclusions in Sec. VII.

II. ZHANG'S ARGUMENT FOR THE NEGATIVE TAIL EXPONENT

A. Distribution of the free energy in the low-temperature phase

According to the droplet theory [3], the whole low-temperature phase $0<T<T_c$ is governed by a zero-temperature fixed point. In particular, at $T<T_c$, the droplet exponent $\theta(d)$ governs the width $\Delta F(L,T)$ and the correction to extensivity of the average $F^{\rm av}(L,T)$

$$\Delta F(L,T) \sim L^{\theta(d)} f_2(T) + \cdots, \tag{12}$$

$$F^{\rm av}(L,T) \sim L f_0(T) + L^{\theta(d)} f_1(T) + \cdots \tag{13}$$

and the rescaled probability distribution of the free energy coincides with the rescaled distribution P_d describing the ground-state energy distribution (1)

$$\mathcal{P}_d(F,L,T) \simeq \frac{1}{\Delta F(L,T)} P_d\left(x = \frac{F - F^{\rm av}(L,T)}{\Delta F(L,T)} \right) \tag{14}$$

as recently checked numerically using simple sampling [28].

B. Zhang's argument for the directed polymer

In finite dimensions $d>1$, the rescaled distribution P_d is not known but there exists a simple argument due to Zhang [12] that allows one to determine the exponent η of the negative tail of the free-energy distribution

$$P_d(x \to -\infty) \sim e^{-c|x|^{\eta(d)}}. \tag{15}$$

If $\eta(d)>0$, the moments of the partition function can be evaluated by the saddle-point method, with a saddle value F^* lying in the negative tail (15)

$$\overline{Z_L^n} = \int dF \mathcal{P}_L(F,L) e^{-n\beta F} \sim \int dF e^{-c(|F|/L^{\theta(d)})^{\eta(d)}} e^{-n\beta F} \sim e^{b(n)L^{\theta(d)\eta(d)/[\eta(d)-1]}}, \quad (16)$$

where $b(n)$ is the prefactor depending on n, but here we focus on the dependence in L. Since for positive integer n, these moments of the partition function can be formulated in terms of the iteration of some transfer matrix, they have to diverge exponentially in L with some Lyapunov exponent. As a consequence, the exponent $\eta(d)$ of the negative tail (15) is not a free parameter, but is fixed by the value of the fluctuation exponent

$$\eta(d) = \frac{1}{1-\theta(d)}. \quad (17)$$

In dimension $d=1$ where the droplet exponent is exactly known $\theta(d=1)=1/3$ [Eq. (9)], this yields the negative tail exponent

$$\eta(d=1) = \frac{3}{2} \quad (18)$$

in agreement with the exact Tracy-Widom distributions [19–21]. In dimensions $d=2$ and $d=3$, the numerical estimates of the droplet exponents [Eq. (11)] yield the following predictions:

$$\eta(d=2) \sim 1.32,$$

$$\eta(d=3) \sim 1.23. \quad (19)$$

These predictions have been tested numerically in Ref. [27] using simple sampling that do not allow one to have data far in the tails. In the following, we will use the importance sampling Monte Carlo method in the disorder to probe the negative tail more precisely.

C. Zhang's argument for spin glasses

To the best of our knowledge, Zhang's argument seems to be applied only in the context of directed polymers [12], whereas it can be applied for other kinds of disordered systems since it is only based on a scaling argument within a saddle-point approximation in the large L limit [Eq. (16)]. It is thus interesting to describe now its implications in the field of spin glasses.

For spin models in finite dimensions, the fluctuations of free energies over the samples scale instead as

$$[\Delta F_L]_{\text{samples}} \sim L^{d/2} \quad (20)$$

at any temperature as proven in Ref. [5]. This scaling simply reflects the central-limit fluctuations of the L^d disorder variables defining the sample. (The directed polymer escapes from these normal fluctuations because it is a one-dimensional path living in a $1+d$ disordered sample: each configuration of the polymer only sees L random variables among the L^{1+d} disorder variables that define the sample, and the polymer can "choose" the random variables it sees.)

Repeating Zhang's argument in this case (20) yields for the negative tail exponent $\eta(d)=2$. This is in agreement with the recent numerical studies [2,11] that find a Gaussian distribution in finite dimensions and in the one dimensional Ising spin glass with long range interactions in the non-mean-field regime.

On the contrary, for the Sherrington-Kirkpatrick model [2,9,10], the probability distribution of the ground state is found to be asymmetric, and has been fitted with generalized Gumbel distribution [9,10]. However, if one repeats Zhang's argument for the SK model with the measured fluctuation exponent $\theta_f \sim 0.235$ [9] for the width $\Delta E_0(N) \sim N^{\theta_f}$, one obtains the negative tail exponent

$$\eta_{\text{SK}} = \frac{1}{1-\theta_f} \sim 1.3. \quad (21)$$

If the value of the fluctuation exponent is exactly $\theta_f = 1/4$ as suggested by some theoretical arguments [2,8], the negative tail exponent would be $\eta_{\text{SK}}=4/3$.

This could explain why the fit with generalized Gumbel distributions whose negative tail is a simple exponential $e^{-m|x|}$ with exponent $\eta=1$ and coefficient m leads to increasing effective values of m when the range over which the tail is measured grows: the fit with simple scaling data on $x \geqslant -6$ leads to $m \sim 6$ [9], whereas the importance-scaling data on $x \geqslant -15$ leads to a completely different estimate $m \sim 11$ [10].

III. DESCRIPTION OF THE IMPORTANCE-SAMPLING MONTE CARLO ALGORITHM IN THE DISORDER

In Ref. [10], a procedure based on an importance-sampling Monte Carlo algorithm in the disorder was proposed to probe with high precision the tails of the ground-state energy distribution of disordered systems, and was applied for the Sherrington-Kirpatrick mean-field Ising spin glass, where probabilities up to 10^{-18} could be measured. In this section, we summarize their method which can be divided in three steps. For each step we mention the specific choices we have made to apply it to the directed polymer model.

A. Simple sampling

A disorder configuration will be denoted by $\mathcal{D}$, and its ground-state energy $E(\mathcal{D})$. For the directed polymer, the ground-state energy can be computed via the transfer matrix. A simple sampling numerical estimation $P_{\text{simple}}(E)$ of the ground-state energy distribution $P(E)$ consists in drawing n_s independent disordered samples $\mathcal{D}_1, \ldots, \mathcal{D}_{n_s}$, in computing the corresponding ground-state energies $E(\mathcal{D}_1), \ldots, E(\mathcal{D}_{n_s})$, and in constructing the histogram

$$P_{\text{simple}}(E) = \frac{1}{n_s} \sum_{i=1}^{n_s} \delta(E - E(\mathcal{D}_i)). \quad (22)$$

This histogram is very useful to measure the distribution $P(E)$ where $P(E) \gg \frac{1}{n_s}$, but gives no information on the tails where $P(E) < \frac{1}{n_s}$, since no events are found.

As an example, we show in Fig. 1 the results we have obtained recently via simple sampling for $d=1,2,3$, respec-

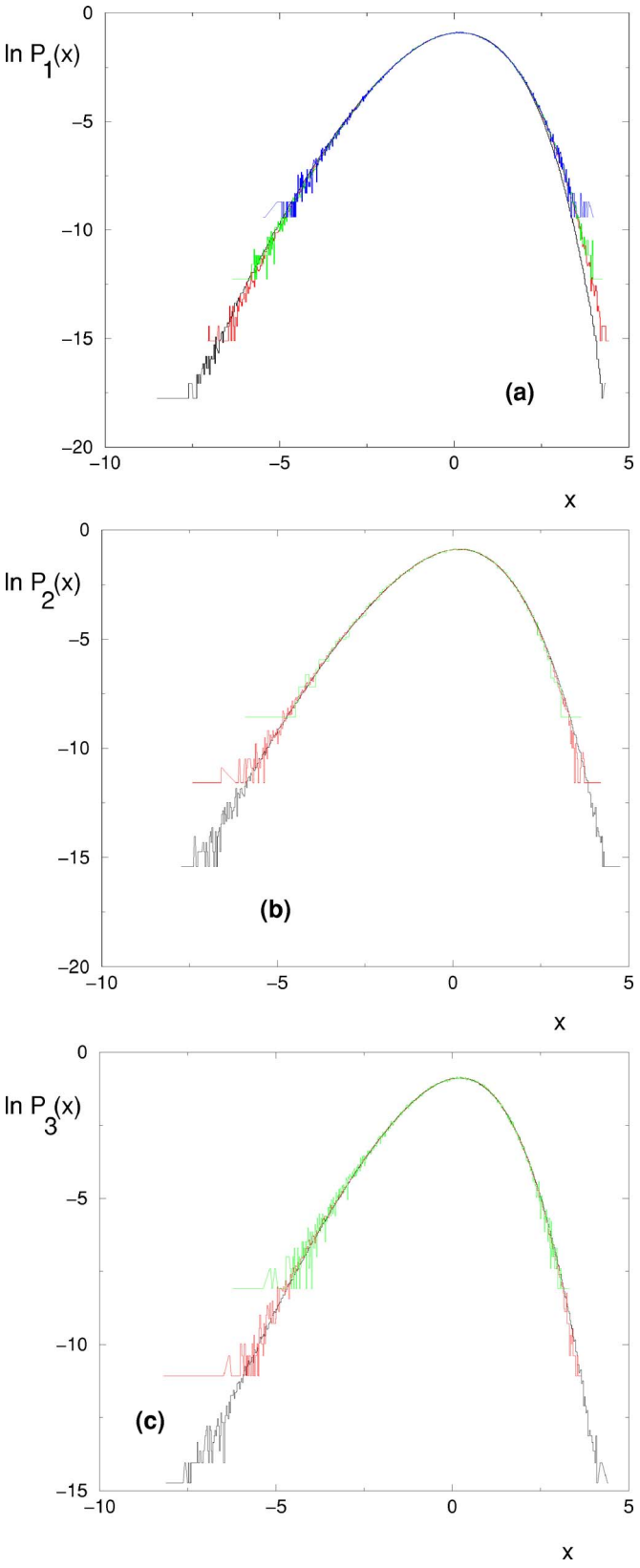

FIG. 1. (Color online) Logarithmic plot of the rescaled probability distribution $P_d(x)$ [Eq. (1)] as measured via simple sampling for (a) $L=12,50,200,800$ in $d=1$; (b) $L=10,40,160$ in $d=2$; (c) $L=6,18,36$ in $d=3$.

tively [16]: whereas the core of the distribution is well measured, the tails suffer from statistic fluctuations as soon as the probability becomes too small. Moreover, if one chooses to make the same CPU effort on all sizes, the number of samples rapidly decays with the size L, so that the data for the tails are less and less precise as L grows. Since one is interested into the asymptotic regime $L \to \infty$, the correct measure of the tails quickly becomes intractable within the simple sampling procedure.

This is why a correct measure of the tails requires the use of some importance sampling, as stressed in Ref. [10]. However, the simple sampling study is the first necessary step within the present method, for three reasons.

(i) The simple sampling results are needed to construct the guiding function of the importance sampling measure [10] as described below.

(ii) The simple sampling results give accurate results for the average value $E_0^{\rm av}(L)$ and the standard deviation $\Delta E_0(L)$, that do not have to be measured via importance sampling [10]. In particular, this allows one to work on a finite box $[x_{\rm min}, x_{\rm max}]$ for the rescaled variable (1), and to choose freely the boundaries of the box, for instance, $x_{\rm max}=-1$ to concentrate on the negative tail, as will be done below for the directed polymer.

(iii) Finally, the simple sampling results allow one to check the validity of the importance-sampling measures on the core of the distribution where the simple sampling results are sufficiently precise.

B. Construction of a guiding function $G(E)$ from the simple sampling result $P_{\rm simple}(E)$

The simple sampling result $P_{\rm simple}(E)$ exists in the range in E where $P_{\rm simple}(E) > 1/n_s$, whereas the guiding function $G(E)$ needed for the importance sampling below has to be defined in the tails where $P_{\rm simple}(E) < 1/n_s$. The guiding function $G(E)$ should be in some sense the "best" extrapolation of the data $P_{\rm simple}(E)$. The proposal of Ref. [10] is to define $G(E)$ as the best fit of $P_{\rm simple}(E)$ within the one parameter family of generalized Gumbel distribution $g_m(x)$, which reads for the normalization conditions $\langle x \rangle = 0$ and $\langle x^2 \rangle = 1$

$$g_m(x) \equiv \frac{1}{\beta(m)} \frac{m^m}{\Gamma(m)} (e^{[x-\alpha(m)]/\beta(m) - e^{[x-\alpha(m)]/\beta(m)}})^m, \quad (23)$$

where $\beta(m) = \frac{1}{\sqrt{\Gamma''(m)/\Gamma(m) - [\Gamma'(m)/\Gamma(m)]^2}}$ and $\alpha(m) = -\beta(m)\left(\frac{\Gamma'(m)}{\Gamma(m)} - \ln m\right)$. The usual Gumbel distribution corresponds to $m=1$, whereas the Gaussian can be formally recovered in the limit $m \to \infty$. This choice was motivated by the numerical finding that the rescaled probability distribution as measured via simple sampling could be fitted extremely well by a generalized Gumbel distribution $g_m(x)$ of parameter $m \sim 6$ [9,10]. It turns out that many recent studies in various contexts have found that asymmetric distributions could be extremely well fitted by generalized Gumbel distributions with various non-integer values of m [29–32]. This has motivated theoretical studies to understand the origin of this type of distribution

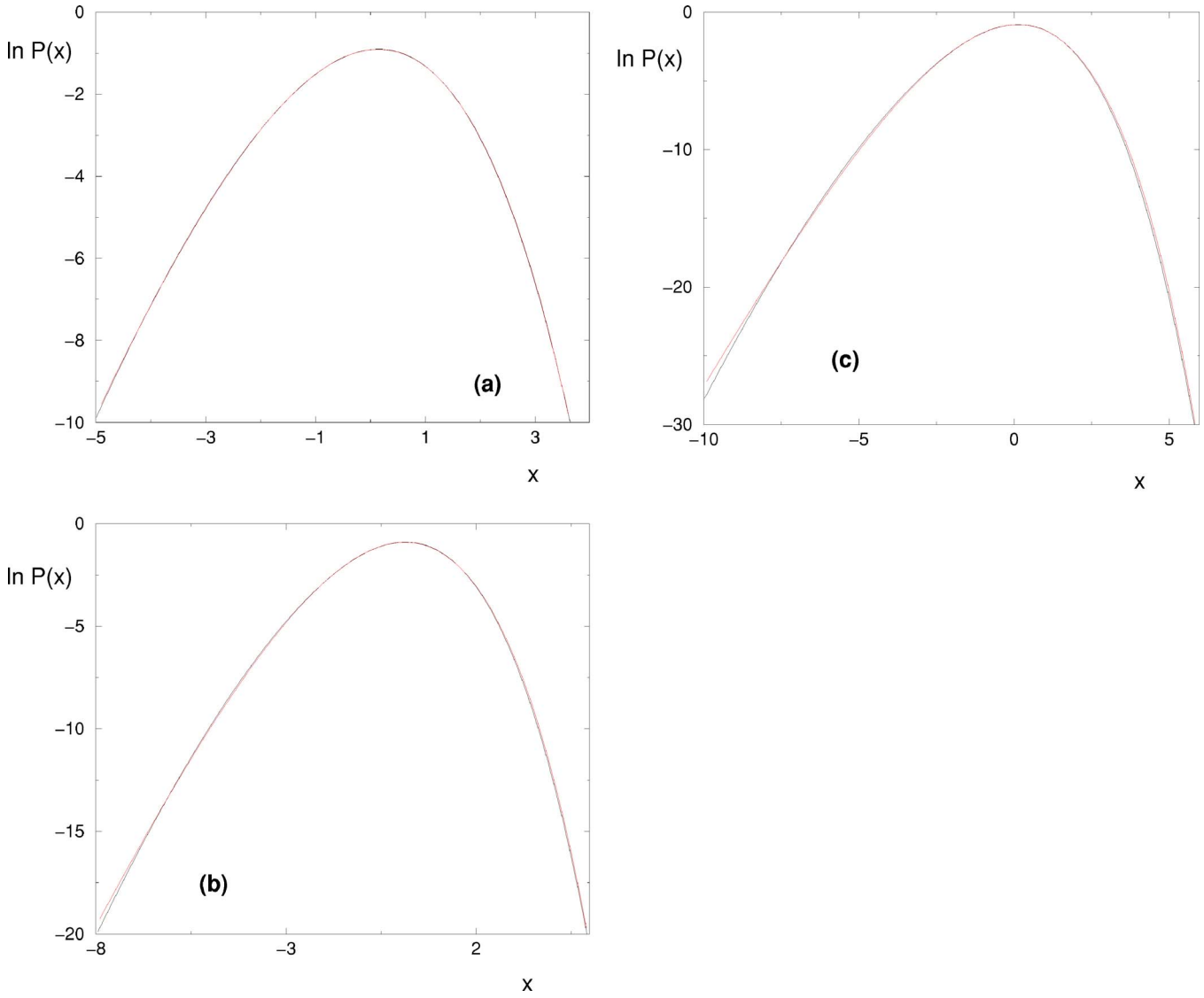

FIG. 2. (Color online) Fits of the GOE Tracy-Widom distribution $\ln P_{\mathrm{TW}}^{\mathrm{GOE}}(x)$ by generalized Gumbel distributions $\ln g_m(x)$ [Eq. (23)] on various intervals: (a) fit on $[-5,3.7]$ corresponding to $\ln P > -10$ with $m=12.93$, (b) fit on $[-8,5]$ corresponding to $\ln P > -20$ with $m=14.71$, (c) fit on $[-10,6]$ corresponding to $\ln P > -30$ with $m=15.92$.

[33]. However as discussed in Ref. [29], it is empirically known that probability distribution functions (PDF) with the same first four moments approximately coincide over the range of a few standard deviations which is precisely the range of numerical—or experimental—data. So generalized Gumbel PDF's, with arbitrary m, should not be considered more than a convenient one parameter fit. To demonstrate clearly how misleading these fits can be, we show in Fig. 2 how the Tracy-Widom GOE distribution $P_{\mathrm{TW}}^{\mathrm{GOE}}$ (which represents the exact rescaled distribution for the directed polymer model in $d=1$ [19–21]) can be fitted by generalized Gumbel distributions on the three ranges $P > 10^{-10}$, $P > 10^{-20}$, and $P > 10^{-30}$: the best fit corresponds to increasing values of the parameter m. For the first range $P > 10^{-10}$, the found fit is "perfect," whereas a slight difference begins to appear in the negative tail as the range grows. Moreover, the Tracy-Widom distribution is known to have the following asymptotic behavior:

$$P_{\mathrm{TW}}^{\mathrm{GOE}}(x) \underset{x \to -\infty}{\simeq} e^{-c_1 |x|^{\eta_1}} \quad \text{with } \eta_1 = \frac{3}{2}, \tag{24}$$

whereas the generalized Gumbel distribution has for any m an exponential tail with exponent $\eta=1$ and coefficient m

$$g_m(x) \underset{x \to -\infty}{\simeq} e^{-m|x|}. \tag{25}$$

This explains why the effective m of the best fit grows with the range. In conclusion, whenever the fit of the core of the distribution leads to an effective m which grows with the range, as in the SK model where $m \sim 6$ and $m \sim 11$ were found depending on the range [9,10], the PDF is probably not a generalized Gumbel distribution, but is likely to have a negative tail exponent $\eta > 1$ [as already suggested around Eq. (21) using Zhang's argument]. And if one focuses on the negative tail, it is clear that the fit with a simple exponential (25) is very restrictive.

As a consequence, in the following where we focus on the negative tail $x \leqslant -1$ for the directed polymer, we have chosen not to work with the generalized Gumbel distribution, but to construct a guiding function $G(E)$ which fits the simple sampling data and whose leading behavior involves the negative tail exponent η_d as obtained from Zhang's argument [see Eqs. (17)–(19)]. In practice, we have found it convenient to work in $d=2$ and $d=3$ on the range $x \in [-10,-1]$ with some guiding function $G_d(x)$ of the form

$$\ln G_d(x) = a_0 - a_1|x|^{\eta_d} + a_2 \ln|x|, \tag{26}$$

where the three parameters $a_i(d)$ were chosen to fit best the simple sampling data.

C. Importance sampling with the guiding function $G(E)$

The importance-sampling Monte Carlo algorithm proposed in Ref. [10] is defined by the following Markov chain.

(1) From the current disorder configuration $\mathcal{D}_i$, construct a candidate $\mathcal{D}'$ for the next disorder configuration $\mathcal{D}_{i+1}$ by replacing a subset of $\mathcal{D}_i$ chosen at random with new values drawn with the original disorder distribution. For a spin model of N spins, this subset can be for instance a single bond chosen at random, or all bonds connected to a site chosen at random, so that the proposed change in the ground state energy is of order $O(1)$ with respect to a value of order $E_0(N)=Ne_0+\cdots$, i.e., its relative order of magnitude is of order $1/N$ [10]. For the directed polymer studied here, we have chosen for this subset the energies of a whole time slice, i.e., all the disorder variables seen by a given monomer. Then the proposed change in the ground-state energy is of order $O(1)$ with respect to a value of order $E_0(N)=Ne_0+\cdots$ as in spin models.

(2) Calculate the new ground-state energy $E(\mathcal{D}')$ and compare it with the previous ground-state energy $E(\mathcal{D}_i)$ using the guiding function $G(E)$: set $\mathcal{D}_{i+1}=\mathcal{D}'$ with probability

$$p_{\text{accept}}(\mathcal{D}'|\mathcal{D}_i) = \min\left[\frac{G(E(\mathcal{D}_i))}{G(E(\mathcal{D}'))},1\right] \tag{27}$$

and set $\mathcal{D}_{i+1}=\mathcal{D}_i$ otherwise.

This Markov chain is expected to converge towards a stationary state where a disorder configuration $\mathcal{D}$ is visited with probability $\propto 1/G(E(\mathcal{D}))$. The stationary probability to visit a disorder configuration with energy E is now given by the ratio

$$R_{\text{stationary}}(E) = \frac{P(E)}{G(E)}. \tag{28}$$

If the guiding function $G(E)$ were the exact $P(E)$, this would correspond to a flat-histogram sampling of $P(E)$. If $G(E)$ is just a reasonable extrapolation of the simple sampling result $P_{\text{simple}}(E)$, one expects to measure nevertheless much better the tails of $P(E)$.

(3) Measurements from the Monte Carlo procedure: Since successive configurations visited by a Monte Carlo algorithm are not independent, one usually keeps only decorrelated configurations for the numerical measure $R_{\text{importance}}(E)$ of the theoretical stationary solution $R_{\text{stationary}}(E)$. This means in practice that one should first estimate some typical correlation time τ and use only every τth configuration

$$R^{(\tau)}_{\text{importance}}(E) = \frac{1}{m_I}\sum_{j=1}^{m_I} \delta(E - E(\mathcal{D}_{i+j\tau})), \tag{29}$$

where the number m_I of measured points is simply the ratio $m_I=\frac{T}{\tau}$ of the total number T of Monte Carlo iterations by the correlation time τ. For instance, in Ref. [10], the time τ was chosen to be $\tau=4\tau_e$ where τ_e is the time where the autocorrelation of the ground-state energy

$$C(t) = \frac{\langle E_i E_{i+t}\rangle - \langle E_i\rangle\langle E_{i+t}\rangle}{\langle E_i^2\rangle - \langle E_i\rangle^2} \tag{30}$$

decays to $1/e$. For the SK model with $16 \leqslant N \leqslant 128$ spins, the autocorrelation time was found to be of order of 400–700 MC steps [10].

For the directed polymer, we actually find that the histograms $R^{(\tau)}_{\text{importance}}(E)$ obtained for $\tau=1$ and $\tau \gg \tau_e$ coincide, except that the histograms with large τ contain more noise since they are built out of less events. From a theoretical point of view, one can justify this finding as follows: if the total Monte Carlo time T is much bigger than the typical time t_{cross} to cross the interval $[E_{\min},E_{\max}]$, then the average with respect to the stationary measure should be equivalent to the time average of the Monte Carlo procedure where all times are kept

$$\int dE f(E) P_{\text{stationary}}(E) = \frac{1}{T}\sum_{t=1}^{T} f(E(t)) \quad \text{for } T \gg t_{\text{cross}}. \tag{31}$$

Indeed for a free random walk in a finite box, it seems clear that one obtains the flat histogram via measuring the positions at all times, instead of throwing away most of the times to have independence between two consecutive measures. The quality of the convergence towards the stationary distribution then depends on the number

$$n_{\text{cross}} \sim \frac{T}{t_{\text{cross}}} \tag{32}$$

of crossings of the interval $[E_{\min},E_{\max}]$ during the total number T of the Monte Carlo, which should be large enough $n_{\text{cross}} \gg 1$.

D. Summary of the procedure used for the directed polymer

In the following sections, we will present the results for the ground-state energy distribution obtained by combining (i) the Monte Carlo procedure in the disorder discussed above and (ii) the transfer matrix calculation of the ground-state energy in each sample [12] with a free boundary condition for the end polymer. We have chosen to focus on the negative tail, by working on the same finite box $x \in [x_{\min},x_{\max}]$ in terms of the rescaled variable x [Eq. (1)] for all sizes L. We now present our results for $d=1,2,3$, respectively.

IV. MEASURE OF THE GROUND-STATE ENERGY DISTRIBUTION IN $d=1$

In $d=1$, the exact rescaled distribution of the ground-state energy is exactly known and corresponds to the Tracy-Widom GOE distribution $P_{\rm TW}^{\rm GOE}$ [19–21] if the last monomer is free, the case we consider here. (It would be the Tracy-Widom GUE distribution $P_{\rm TW}^{\rm GUE}$ if the last monomer were fixed at the origin.) We use this exact result to check the validity of the Monte Carlo procedure in the disorder and to describe its main properties.

A. Numerical details

In dimension $d=1$, we have chosen to work on the interval $[x_{\rm min},x_{\rm max}]=[-11.0,-1.0]$ for the rescaled variable x [Eq. (1)], i.e., to probe the negative tail up to probabilities of order $P_1(x)>10^{-32}$. We now give the sizes L we have studied, together with the standard deviation $\Delta E_0(L)$ measured by simple sampling and used in the rescaling of Eq. (1) [the averaged values $E_0^{\rm av}(L)$ can be found in our previous work [16]], the corresponding number T_L of Monte Carlo iterations, the acceptance rate $\tau_{\rm acc}(L)$ of Monte Carlo moves, and the number $n_{\rm cross}(L)$ of crossings of the box $[x_{\rm min},x_{\rm max}]=[-11.0,-1.0]$.

$$L=50,100,200,400,800,1600, \tag{33}$$

$$\Delta E_0(L)\sim 3.12,3.98,5.04,6.36,8.04,10.11, \tag{34}$$

$$T_L=33\times 10^8,85\times 10^7,225\times 10^6,57\times 10^6, 7\times 10^6,4\times 10^6, \tag{35}$$

$$\tau_{\rm acc}(L)\sim 0.54,0.64,0.72,0.76,0.82,0.86, \tag{36}$$

$$n_{\rm cross}(L)\sim 98\times 10^4,224\times 10^3,45\times 10^3,8\times 10^3,670,230. \tag{37}$$

As L grows, the proposed Monte Carlo moves Δx in the rescaled variable x [Eq. (1)] are smaller: this is why both the acceptance rate $\tau_{\rm acc}(L)$ and the crossing time $t_{\rm cross}(L)$ [Eq. (32)] also grows with L. The final result is that the number of crossing $n_{\rm cross}(L)$ decays with L, and since it should remain large enough to obtain a good measure [Eq. (32)], this number fixes the maximal size that can be correctly studied.

B. Properties of the Monte Carlo process in the disorder

We show in Fig. 3 the histograms of the proposed and accepted Monte Carlo changes in the disorder, for $L=25$ and for $L=1600$, respectively. The proposed changes are biased towards $\Delta x>0$, because here, in the negative tail, a Monte Carlo step $\Delta x>0$ corresponds to a move where the probability $P(x)$ is bigger. The histogram of the accepted moves is on the contrary almost symmetric around $\Delta x=0$ in order to generate a nonbiased random walk. For $\Delta x<0$, the two histograms almost coincide, i.e., a move $\Delta x<0$ is almost always accepted. As the size L grows, the proposed moves in the relative variable x are smaller, and as a consequence, the acceptance rate grows with L.

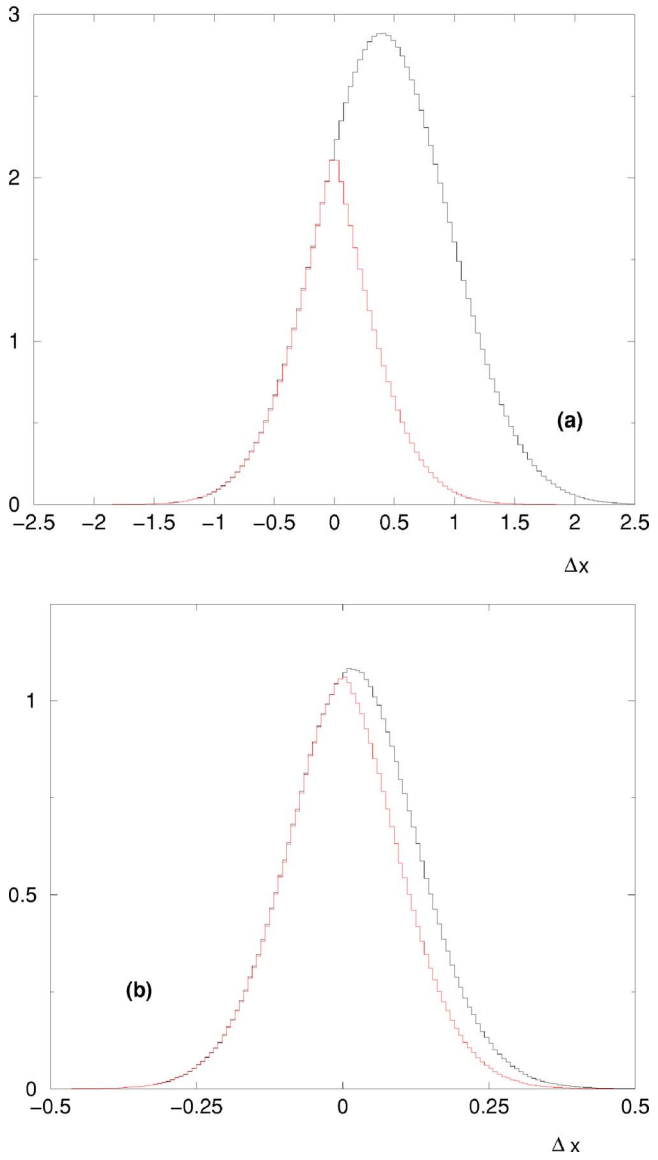

FIG. 3. (Color online) Monte Carlo procedure to measure the negative tail on $x\in[-11,-1]$ in $d=1$: histograms of the proposed and accepted Monte Carlo changes Δx in the disorder (a) for $L=25$ where the acceptance rate is $\tau_{\rm acc}\sim 0.425$ and (b) for $L=1600$ where the acceptance rate is $\tau_{\rm acc}\sim 0.86$.

The resulting process $x(t)$ are shown on Fig. 4 for the first $1\leqslant t\leqslant 10\,000$ Monte Carlo iterations, for $L=50$ and $L=200$, respectively. The time $t_{\rm cross}$ needed to cross the interval $[x_{\rm min},x_{\rm max}]=[-11,-1]$ grows with L.

C. Convergence towards the exact Tracy-Widom distribution

In Fig. 5(a), we show the relative histogram $P_L(x)/P_{\rm TW}^{\rm GOE}(x)$ of the measured $P_L(x)$ via the Monte Carlo procedure as L grows with respect to the Tracy-Widom GOE distribution that represents the asymptotic exact result for $L\to\infty$: these relative histograms become flatter as L grows. In Fig. 5(b), we show for comparison (i) the simple sampling

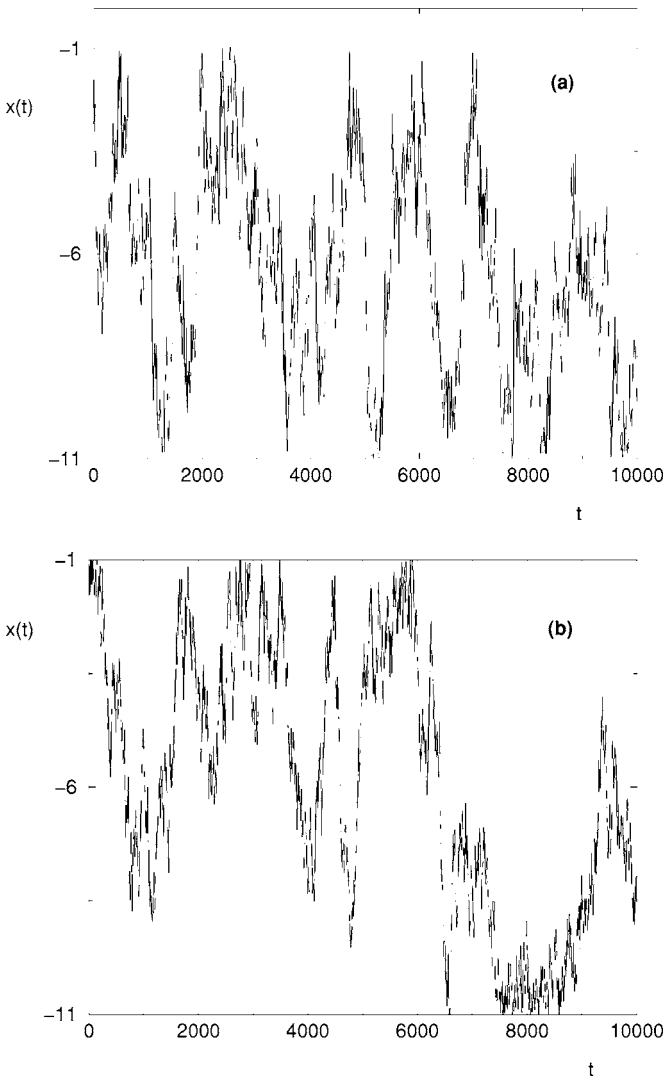

FIG. 4. Monte Carlo procedure to measure the negative tail on $x \in [-11,-1]$ in $d=1$: process $x(t)$ during the first $1 \leqslant t \leqslant 10\,000$ Monte Carlo iterations (a) for $L=50$, (b) for $L=200$.

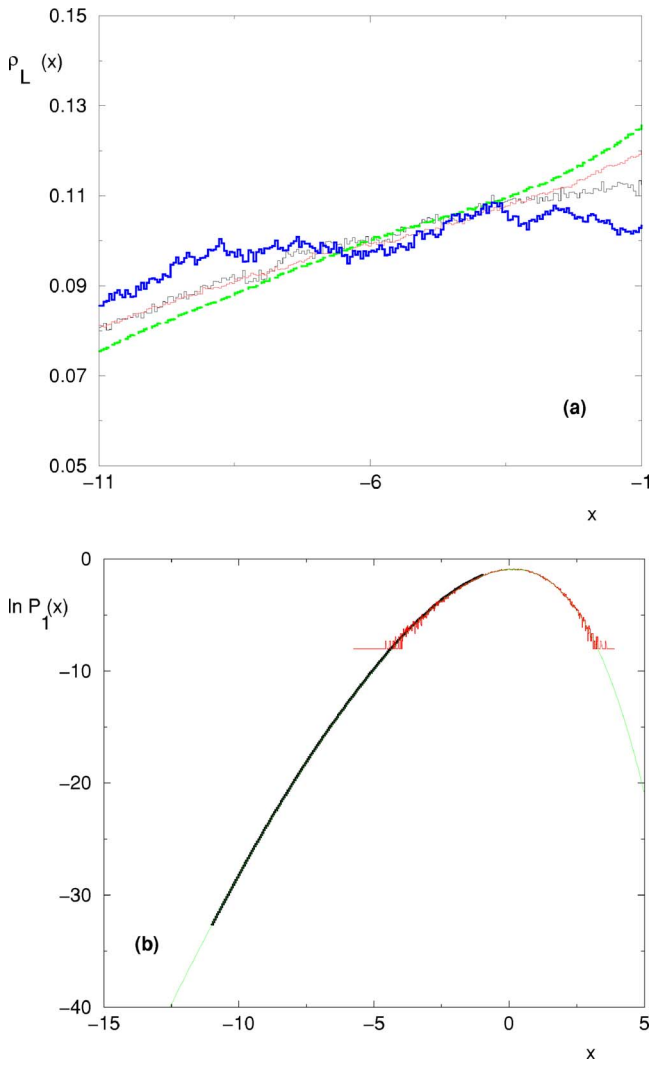

FIG. 5. (Color online) Monte Carlo procedure to measure the negative tail on $x \in [-11,-1]$ in $d=1$: (a) relative histogram $\rho_L(x) = P_L(x)/P_{\mathrm{TW}}^{\mathrm{GOE}}(x)$ with respect to the exact guiding function: convergence towards the flat histogram as L grows: $L=200$ (dashed line), 400, 800, 1600 (thick line). (b) Logarithmic plot of the negative tail of the probability distribution $P_1(x)$, as compared to simple sampling result for $L=1600$. The exact Tracy-Widom distribution is also shown (thin line) to demonstrate the validity of the Monte Carlo procedure.

histogram for $L=1600$, (ii) the importance-sampling measure of the tail on $x \in [-11,-1]$ for $L=1600$, and (iii) the exact Tracy-Widom GOE distribution. Our conclusion is thus that the Monte Carlo in the disorder is a very efficient method to probe accurately the tails, since they allow us to reproduce the exact result on the range $x \in [-11,-1]$ for sizes up to $L=1600$.

D. Extraction of the negative tail exponent value

Let us now make some comments on the extraction of the negative tail exponent value. The exact Tracy-Widom GOE distribution $P_{\mathrm{TW}}^{\mathrm{GOE}}(x)$ has for negative exponent $\eta_1=3/2$. However, the following fits of this distribution $P_{\mathrm{TW}}^{\mathrm{GOE}}(x)$ on the finite range $x \in [-11,-1]$ give slightly larger values: (i) the fit of $[\ln P_{\mathrm{TW}}^{\mathrm{GOE}}(x)]$ by $a-b(-x)^{\eta_1}$ containing three parameters yields $\eta_1 \sim 1.58$ and (ii) the fit of $[\ln P_{\mathrm{TW}}^{\mathrm{GOE}}(x)]$ by $a-b(-x)^{\eta_1}+c\ln(-x)$ containing four parameters yields $\eta_1 \sim 1.54$. This shows that the extracted value of the negative tail exponent from data on the finite range $x \in [-11,-1]$ is not very precise if there is no information on the subleading terms. Similarly in higher d below, we expect that the Monte Carlo procedure gives very accurate data on the range where the tail is measured, but that the extraction of the negative tail exponent value suffers from some error directly related to the range that is probed.

V. RESULTS FOR THE GROUND-STATE ENERGY DISTRIBUTION IN $d=2$

A. Numerical details

In dimension $d=2$, we have chosen to work on the interval $[x_{\min},x_{\max}]=[-10.0,-1.0]$ for the rescaled variable x [Eq.

(1)], i.e., to probe the negative tail up to probabilities of order $P_1(x) > 10^{-23}$. We now give the sizes L we have studied, together with the standard deviation $\Delta E_0(L)$ measured by simple sampling and used in the rescaling of Eq. (1) [the averaged values $E_0^{\mathrm{av}}(L)$ can be found in our previous work [16]], the corresponding number T_L of Monte Carlo iterations, the acceptance rate $\tau_{\mathrm{acc}}(L)$ of Monte Carlo moves

$$L = 20,40,80,120,160, \tag{38}$$

$$\Delta E_0(L) \sim 1.58,1.85,2.18,2.40,2.54, \tag{39}$$

$$T_L = 125 \times 10^7, 27 \times 10^7, 47 \times 10^6, 34 \times 10^5, 64 \times 10^4, 21 \times 10^4, \tag{40}$$

$$\tau_{\mathrm{acc}}(L) \sim 0.24,0.28,0.38,0.47,0.5,0.54. \tag{41}$$

B. Monte Carlo results

In Fig. 6(a), we show the process $x(t)$ during the first 10 000 Monte Carlo iterations for $L=120$. In Fig. 6(b), we compare the importance-sampling measure of the negative tail with respect to the simple sampling evaluation.

C. Negative tail exponent $\eta_{d=2}$

From the point of view of the convergence in L towards a fixed distribution, we find that the negative tail measured for the two bigger sizes $L=120$ and $L=160$ nearly coincide on the whole interval $[x_{\min},x_{\max}]=[-10.0,-1.0]$ under study (whereas our results for the smaller sizes do not).

As explained previously in Sec. IV D for the case $d=1$, the error on the estimated value of the negative tail exponent is due to the range $[x_{\min},x_{\max}]=[-10.0,-1.0]$ over which the fits are made. As in Sec. IV D, we have tried to fit our result for $[\ln P_2(x)]$ as measured for the sizes $L=120$ and $L=160$ by the two following fits, with or without power-law corrections with respect to the leading exponential term (i) the first fit $a-b(-x)^{\eta_2}$ containing three parameters yields $\eta_2 \sim 1.4$ and (ii) the second fit by $a-b(-x)^{\eta_1}+c\ln(-x)$ containing four parameters yields $\eta_2 \sim 1.3$. Our conclusion is thus that the extracted value of the negative tail exponent from our data on the finite range $x \in [-10,-1]$ is not very precise in the absence of information on the subleading terms, but is compatible with the value $\eta_2^Z = 1.32$ predicted by Zhang's argument [see Eqs. (17) and (19)].

VI. RESULTS FOR THE GROUND-STATE ENERGY DISTRIBUTION IN $d=3$

A. Numerical details

In dimension $d=3$, we have chosen to work on the interval $[x_{\min},x_{\max}]=[-10.0,-1.0]$ for the rescaled variable x [Eq. (1)], i.e., to probe the negative tail up to probabilities of order $P_1(x) > 10^{-21}$. We now give the sizes L we have studied, together with the standard deviation $\Delta E_0(L)$ measured by simple sampling and used in the rescaling of Eq. (1) [the averaged values $E_0^{\mathrm{av}}(L)$ can be found in our previous work [16]], the corresponding number T_L of Monte Carlo iterations, the acceptance rate $\tau_{\mathrm{acc}}(L)$ of Monte Carlo moves, and the number $n_{\mathrm{cross}}(L)$ of crossings of the box $[x_{\min},x_{\max}]=[-10.0,-1.0]$.

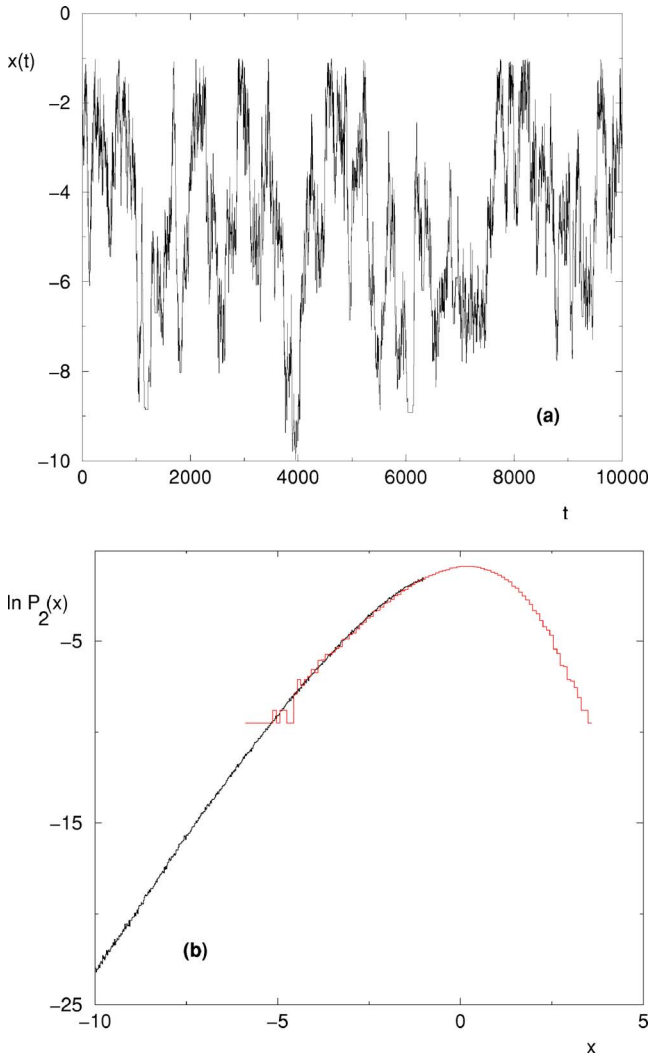

FIG. 6. (Color online) Monte Carlo procedure to measure the negative tail on $x \in [-10,-1]$ in $d=2$ for $L=120$: (a) process $x(t)$ during the first $1 \leqslant t \leqslant 10\,000$ Monte Carlo iterations, (b) logarithmic plot of the negative tail of the probability distribution $P_2(x)$, as compared to a simple sampling result.

$$L = 12,24,36,48,60,72, \tag{42}$$

$$\Delta E_0(L) \sim 1.15,1.30,1.39,1.46,1.52,1.55, \tag{43}$$

$$T_L = 64 \times 10^6, 43 \times 10^5, 75 \times 10^4, 95 \times 10^4, 34 \times 10^4, 182 \times 10^3, \tag{44}$$

$$\tau_{\mathrm{acc}}(L) \sim 0.24,0.27,0.32,0.34,0.36,0.37, \tag{45}$$

$$n_{\mathrm{cross}}(L) \sim 20\,000,2400,300,522,166,94. \tag{46}$$

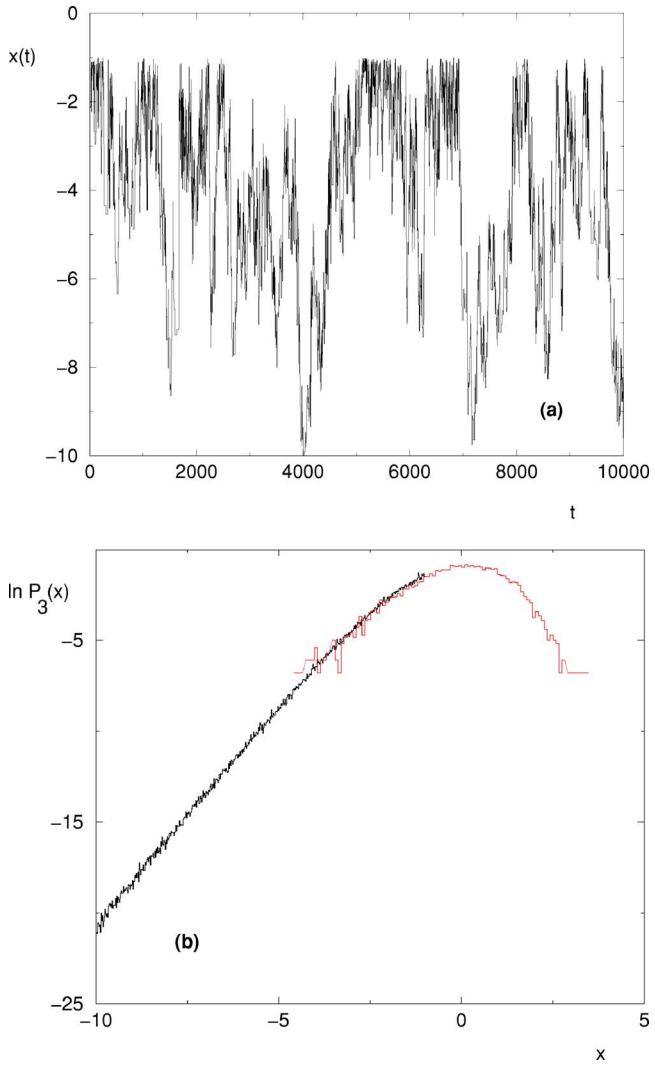

FIG. 7. (Color online) Monte Carlo procedure to measure the negative tail on $x \in [-10,-1]$ in $d=3$ for $L=72$: (a) process $x(t)$ during the first $1 \leqslant t \leqslant 10\,000$ Monte Carlo iterations, (b) logarithmic plot of the negative tail of the probability distribution $P_3(x)$, as compared to simple sampling result.

B. Monte Carlo results

In Fig. 7(a), we show the process $x(t)$ during the first 10 000 Monte Carlo iterations for $L=72$. In Fig. 7(b), we compare the importance sampling measure of the negative tail with respect to the simple sampling evaluation.

C. Negative tail exponent $\eta_{d=3}$

As in Sec. IV D, we have tried to fit our result for $[\ln P_3(x)]$ by the two following fits, with or without power-law corrections with respect to the leading exponential term: (i) the first fit $a-b(-x)^{\eta_3}$ containing three parameters yields $\eta_3 \sim 1.25$ and (ii) the second fit by $a-b(-x)^{\eta_1}+c\ln(-x)$ containing four parameters yields $\eta_3 \sim 1.15$.

Our conclusion is that the extracted value of the negative tail exponent from our data on the finite range $x \in [-10,-1]$ is not very precise in the absence of information on the subleading terms, but is compatible with the value $\eta_3^Z=1.23$ predicted by Zhang's argument [see Eqs. (17) and (19)].

VII. CONCLUSION

In this paper, we have adapted the importance-sampling method in the disorder proposed in Ref. [10] for spin glasses, to measure with high precision the negative tail of the ground-state energy distribution $P_d(E_0)$ for the directed polymer in a random medium of dimension $d=1,2,3$. In $d=1$, we have checked the validity of the procedure by a direct comparison with the exact result, namely, the Tracy-Widom GOE distribution. In dimensions $d=2$ and $d=3$, we have measured the negative tail up to $P \sim 10^{-22}$. Our results are in agreement with Zhang's argument, stating that the negative tail exponent $\eta(d)$ of the asymptotic behavior $\ln P(E_0) \sim -|E_0|^{\eta(d)}$ as $E_0 \to -\infty$ is directly related to the fluctuation exponent $\theta(d)$ via the simple formula $\eta(d)=1/[1-\theta(d)]$.

Along the paper, we have also discussed the similarities and differences with spin glasses. In particular, we have argued that the application of Zhang's argument for the Sherrington-Kirpatrick model of spin glasses points towards an asymptotic distribution which is not a generalized Gumbel distribution $g_m(x)$, in contrast with the current way of fitting the numerical data [9,10], but involves instead some nontrivial negative tail exponent $\eta_{SK}>1$ directly related to the fluctuation exponent [Eq. (21)]. The fact that the fitting value m of generalized Gumbel distribution $g_m(x)$ depends on the probed range in the variable x ($m \sim 6$ via simple sampling [9] and $m \sim 11$ via importance sampling [10]) also points towards $\eta_{SK}>1$. More generally, we have explained in details how fits with generalized Gumbel distributions of the core of the distribution could be very misleading if one is interested in the tails, since all Gumbel distributions correspond to the exponent $\eta=1$, which is very restrictive.

Finally, our conclusion concerning the algorithm is that the importance-sampling Monte Carlo Markov chain in the disorder introduced in Ref. [10] is a very efficient method to probe precisely the tails of probability distributions over the samples. In the field of disordered systems, this Monte Carlo procedure will be very useful to study probability distributions of other observables in addition to the ground-state energy. Note that it has already been used in the fields of sequence alignments [34] and random graphs [35].

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